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Journal of Algebra 308 (2007) 368–382

JOURNAL OF Algebra

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On a special class of simplicial toric varieties

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Received 13 February 2006

Available online 30 May 2006

Communicated by Luchezar L. Avramov

Dedicated to Professor Winfried Bruns on the occasion of his 60th birthday

Abstract

We show that for all $n \geqslant 3$ and all primes p there are infinitely many simplicial toric varieties of codimension n in the $2n$-dimensional affine space whose minimum number of defining equations is equal to n in characteristic p, and lies between $2n-2$ and $2n$ in all other characteristics. In particular, these are new examples of varieties which are set-theoretic complete intersections only in one positive characteristic. Moreover, we show that the minimum number of binomial equations which define these varieties in all characteristics is 4 for $n=3$ and $2n-2+\binom{n-2}{2}$ whenever $n \geqslant 4$.
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Keywords: Toric variety; Set-theoretic complete intersection; Arithmetical rank; Étale cohomology

0. Introduction

Let K be an algebraically closed field, and let V be an affine variety in K^N. The *arithmetical rank* (ara) of V is defined as the least number of equations that are needed to define V set-theoretically as a subvariety of K^N. In general we have that $\operatorname{ara} V \geqslant \operatorname{codim} V$. If equality holds, V is called a *set-theoretic complete intersection*. In general, the arithmetical rank of a variety may depend upon the characteristic of the ground field: but not many examples of this kind are known so far. The first ones to be found were the determinantal varieties of a symmetric matrix considered in [1]. The paper [3] presents an infinite class of simplicial toric varieties

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[1] Partially supported by the Italian Ministry of Education, University and Research.

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doi:10.1016/j.jalgebra.2006.04.031

of codimension 2 which are set-theoretic complete intersections only in one positive characteristic. The same property has been shown in [2] for the Veronese varieties whose degree is a prime power. These have arbitrarily high codimensions, but only the codimensions of a special form are represented; moreover, in each possible codimension there are only finitely many examples.

In the present paper we will show that for every prime p and every codimension $n \geqslant 3$, there are infinitely many examples of simplicial toric varieties in K^{2n} which are set-theoretic complete intersections (i.e., set-theoretically defined by n equations) only in characteristic p. This result completes the one in [3], but with an interesting difference: as it was proven in [4], the simplicial toric varieties of codimension 2 are always (regardless of their dimension) set-theoretically defined by 3 equations, whereas the arithmetical rank of the varieties that we are going to introduce here dramatically increases from n to a number lying between $2n-2$ and $2n$ when the characteristic is different from p.

It is well known that the defining ideal of every toric variety is generated by binomials. According to a definition introduced by Thoma in [14], the *binomial arithmetical rank* (bar) of a toric variety V is the least number of binomial equations which are needed to define V. Obviously $\operatorname{ara} V \leqslant \operatorname{bar} V$. A complete characterization of the cases where equality holds is given in [5]. For the varieties V that will be presented in this paper, we show that equality holds only for the single positive characteristic where V is a set-theoretic complete intersection; in this case $\operatorname{bar} V = \operatorname{codim} V$, and V is therefore called a *binomial set-theoretic complete intersection*. In the remaining characteristics, $\operatorname{ara} V$ and $\operatorname{bar} V$ differ in a substantial way, since we have that $\operatorname{bar} V = 4$ if $n = 3$, and $\operatorname{bar} V = 2n - 2 + \binom{n-2}{2}$ for all $n \geqslant 4$. Thus we have new examples of affine toric varieties of any codimension greater than or equal to 3 such that, in all but one positive characteristic, the minimum number of defining equations cannot be attained by systems of binomial equations. This property is known to be true, in characteristic zero, for certain projective toric curves in $\mathbf{P}^3_K$, among which the famous Macaulay's curve (see [13]); other special classes of projective toric varieties fulfilling this property in all characteristics have been recently presented in [9, Section 5] and in [8, Sections 5–6].

1. Preliminaries

A *monomial* in a polynomial ring is a product of indeterminates. Given a monomial M, we define the *support* of M, denoted $\operatorname{supp}(M)$, as the set of indeterminates which divide M.

A *binomial* is the difference of two distinct monomials M, M': these will be called the monomials of the binomial $B = M - M'$. This binomial is called *monic* in the indeterminate z if $\operatorname{supp}(M) = \{z\}$ or $\operatorname{supp}(M') = \{z\}$.

Let $n \geqslant 3$ be an integer. Moreover, let $b_1, \ldots, b_{n-2}$ be nonnegative integers, p be a prime, and $\ell, a, d, c_1, \ldots, c_{n-2}$ be positive integers such that

(I) p does not divide any of the c_i;
(II) a and d are coprime;
(III) there are positive integers g, h such that $p^\ell = ag + dh$.

Consider the affine *simplicial toric* variety $V \subset K^{2n}$ admitting the following parametrization:

$$V\colon\quad \begin{cases} x_1 = u_1, \\ \vdots \\ x_{n-2} = u_{n-2}, \\ x_{n-1} = u_{n-1}^{p^\ell}, \\ x_n = u_n^a, \\ y_1 = u_1^{b_1} u_{n-1}^{c_1}, \\ \vdots \\ y_{n-2} = u_{n-2}^{b_{n-2}} u_{n-1}^{c_{n-2}}, \\ y_{n-1} = u_n^d, \\ y_n = u_{n-1} u_n. \end{cases}$$

We have that $\operatorname{codim} V = n$. Let $I(V)$ be the defining ideal of V in the polynomial ring $R = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Ideal $I(V)$ is prime and is generated by binomials. The binomials in $I(V)$ are those of the form:

$$x_1^{\alpha_1^+} \cdots x_n^{\alpha_n^+} y_1^{\beta_1^+} \cdots y_n^{\beta_n^+} - x_1^{\alpha_1^-} \cdots x_n^{\alpha_n^-} y_1^{\beta_1^-} \cdots y_n^{\beta_n^-},$$

where $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \in \mathbb{Z}^{2n} \setminus \{\mathbf{0}\}$ is such that

$$\begin{aligned} &\alpha_1 \mathbf{e}_1 + \cdots + \alpha_{n-1} p^\ell \mathbf{e}_{n-1} + \alpha_n a \mathbf{e}_n \\ &\quad + \beta_1 (b_1 \mathbf{e}_1 + c_1 \mathbf{e}_{n-1}) + \cdots + \beta_{n-2} (b_{n-2} \mathbf{e}_{n-2} + c_{n-2} \mathbf{e}_{n-1}) \\ &\quad + \beta_{n-1} d \mathbf{e}_n + \beta_n (\mathbf{e}_{n-1} + \mathbf{e}_n) = \mathbf{0}, \end{aligned}$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis of $\mathbb{Z}^n$, and we have set $\alpha_i^+ = \max\{\alpha_i, 0\}$, $\alpha_i^- = \max\{-\alpha_i, 0\}$, $\beta_i^+ = \max\{\beta_i, 0\}$ and $\beta_i^- = \max\{-\beta_i, 0\}$.

The next result will be useful in one of the proofs of Section 3.

Lemma 1. *Let B be a binomial of $I(V)$, and let M, M' be its monomials. Then the following conditions hold.*

(i) *For all $i = 1, \ldots, n-2$, x_i divides M if and only if y_i divides M'.*
(ii) *If one of the indeterminates x_n, y_{n-1}, y_n divides M, then one of the remaining two divides M'.*
(iii) *If x_{n-1} divides M, then one of $y_1, \ldots, y_{n-2}, y_n$ divides M'.*

Proof. We prove (i). Let i be any index with $1 \leqslant i \leqslant n-2$. Let $u_i = 0$ and $u_k = 1$ for $k \neq i$. Let $\mathbf{x} = (\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_n) \in V$ be the point corresponding to this choice of parameters. Then

$$\bar{x}_i = \bar{y}_i = 0 \quad \text{and} \quad \bar{x}_k = \bar{y}_k = 1 \quad \text{for all } k \neq i. \tag{1}$$

Since $B \in I(V)$, we have that $B(\mathbf{x}) = 0$. Suppose that x_i divides M. Then $M(\mathbf{x}) = 0$, so that $M'(\mathbf{x}) = 0$. On the other hand, by irreducibility, x_i does not divide M'. In view of (1) it follows that y_i divides M'. The proof of the converse is identical.

Claims (ii) and (iii) can be shown by similar arguments, by selecting certain points of V and using the fact that monomial B vanishes at these points. For the proof of (ii) take $u_n = 0$ and $u_k = 1$ for all $k \neq n$, for the proof of (iii) take $u_{n-1} = 0$ and $u_k = 1$ for all $k \neq n-1$.

Consider the following binomials of R:

$$F_i = y_i^{p^\ell} - x_i^{p^\ell b_i} x_{n-1}^{c_i} \quad (i = 1, \ldots, n-2), \tag{2}$$

$$F_{n-1} = y_{n-1}^a - x_n^d, \tag{3}$$

$$F_n = y_n^{p^\ell} - x_{n-1} x_n^g y_{n-1}^h. \tag{4}$$

In view of (III) it easily follows that $F_1, \ldots, F_n \in I(V)$. □

The next result can be shown by the same arguments as Lemma 1. Nevertheless, we give its proof for the sake of completeness.

Lemma 2. *For all $i = 1, \ldots, n-1$, F_i is the only irreducible binomial in $I(V)$ that is monic in y_i.*

Proof. Consider an index i with $1 \leqslant i \leqslant n-2$. Let $u_i = u_{n-1} = 1$ and $u_k = 0$ for all $k \neq i, n-1$, and let $\mathbf{x} = (\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_n)$ be the corresponding point of V. Then

$$\bar{x}_k = 0 \quad \text{for all } k \neq i, n-1 \quad \text{and} \quad \bar{y}_k = 0 \quad \text{for all } k \neq i. \tag{5}$$

Let $B = M - M' \in I(V)$, where M and M' are monomials, and suppose that $M = y_i^{\alpha_i}$, with $\alpha_i > 0$. Then $M(\mathbf{x}) = 1$, so that, being $B(\mathbf{x}) = 0$, we have $M'(\mathbf{x}) = 1$. In view of (5) and irreducibility, this implies that $\operatorname{supp}(M') \subset \{x_i, x_{n-1}\}$. So let $M' = x_i^{\beta_i} x_{n-1}^{\gamma_i}$. Then

$$\alpha_i b_i = \beta_i, \tag{6}$$

$$\alpha_i c_i = \gamma_i p^\ell. \tag{7}$$

From (7) and (I) we deduce that p^ℓ divides α_i; let $\alpha_i' = \frac{\alpha_i}{p^\ell}$. Then, by (7), $\alpha_i' p^\ell c_i = \gamma_i p^\ell$, whence $\alpha_i' c_i = \gamma_i$. Thus, by (6),

$$B = y_i^{\alpha_i' p^\ell} - x_i^{\alpha_i' p^\ell b_i} x_{n-1}^{\alpha_i' c_i}.$$

But irreducibility implies that $\alpha_i' = 1$. Hence $B = F_i$, as required. The proof for $i = n-1$ is analogous: it suffices to consider the point of V corresponding to $u_n = 1$ and $u_k = 0$ for $k \neq n$ and to argue as above. □

2. The defining equations

In this section we will explicitly give the defining binomial equations for a variety V. We will have to distinguish two cases, according to the characteristic of the ground field K.

Proposition 1. *Suppose that* char $K = p$. *Then* V *is set-theoretically defined by*

$$F_1 = \cdots = F_n = 0.$$

Proof. We only have to prove that every $\mathbf{x} \in K^{2n}$ fulfilling the given equations belongs to V. So let $\mathbf{x} = (\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_n) \in K^{2n}$ be such that $F_i(\mathbf{x}) = 0$ for all $i = 1, \dots, n$. Set

$$u_i = \bar{x}_i \quad \text{for } i = 1, \dots, n-2, \tag{8}$$

and, moreover, let $u_{n-1}, v_n \in K$ be such that

$$\bar{x}_{n-1} = u_{n-1}^{p^\ell} \tag{9}$$

and $\bar{x}_n = v_n^a$. We show that up to replacing v_n with another ath root u_n of $\bar{x}_n$ in K, $\mathbf{x}$ fulfills the parametrization of V given above. Condition $F_{n-1}(\mathbf{x}) = 0$ implies that $\bar{y}_{n-1}^a = v_n^{ad}$, i.e., $\bar{y}_{n-1} = v_n^d \omega$ for some $\omega \in K$ such that $\omega^a = 1$. By virtue of (II) there are integers r, s such that $1 = ar + ds$. Set $\eta = \omega^s$. Then $\eta^a = 1$ and $\eta^d = \omega^{ds} = \omega^{ar+ds} = \omega$. Put $u_n = v_n \eta$. Then

$$\bar{x}_n = v_n^a = v_n^a \eta^a = u_n^a \tag{10}$$

and

$$\bar{y}_{n-1} = v_n^d \omega = (v_n \eta)^d = u_n^d. \tag{11}$$

Furthermore, for all $i = 1, \dots, n-2$, $F_i(\mathbf{x}) = 0$ implies that $\bar{y}_i^{p^\ell} = u_i^{p^\ell b_i} u_{n-1}^{p^\ell c_i}$, i.e.,

$$\bar{y}_i = u_i^{b_i} u_{n-1}^{c_i}. \tag{12}$$

Finally, in view of (III), $F_n(\mathbf{x}) = 0$ implies that $\bar{y}_n^{p^\ell} = u_{n-1}^{p^\ell} u_n^{ag} u_n^{dh} = u_{n-1}^{p^\ell} u_n^{p^\ell}$, i.e.,

$$\bar{y}_n = u_{n-1} u_n. \tag{13}$$

From (8)–(13) it follows that $\mathbf{x}$ fulfills the required parametrization. This completes the proof. □

We have thus proven that V is set-theoretically defined by n binomial equations, i.e., we have the following

Corollary 1. *If* char $K = p$, *the variety* V *is a binomial set-theoretic complete intersection.*

We will show that the above corollary does not extend to the characteristics different from p. In general $F_1, \dots, F_n$ do not suffice to define V set-theoretically: more binomial equations are needed. We are going to define these first. By virtue of (I), for all indices i, j with $1 \leqslant i < j \leqslant n-2$, there are positive integers g_{ij}, not divisible by p, such that

$$c_i g_{ij} + c_j g_{ji} = r_{ij} p^{m_{ij}} \tag{14}$$

for suitable positive integers r_{ij}, m_{ij}. Note that g_{ij} and g_{ji} can be chosen large enough so as to have $m_{ij} \geqslant \ell$. By (I), for all indices i, $1 \leqslant i \leqslant n-2$, there are also positive integers h_i, k_i, not divisible by p, such that

$$c_i h_i + k_i = r_i p^{m_i} \tag{15}$$

for suitable positive integers r_i, m_i. Note that h_i and k_i can be chosen large enough so as to have $m_i \geqslant \ell$. By (II) there are integers s_i, t_i such that

$$a s_i + d t_i = k_i. \tag{16}$$

Up to replacing h_i, k_i with larger numbers we may assume that s_i, t_i are nonnegative. Put

$$G_{ij} = y_i^{g_{ij}} y_j^{g_{ji}} - x_i^{b_i g_{ij}} x_j^{b_j g_{ji}} x_{n-1}^{r_{ij} p^{m_{ij}-\ell}} \quad (1 \leqslant i < j \leqslant n-2),$$

and

$$H_i = y_i^{h_i} y_n^{k_i} - x_i^{b_i h_i} x_{n-1}^{r_i p^{m_i - \ell}} x_n^{s_i} y_{n-1}^{t_i} \quad (i = 1, \ldots, n-2).$$

From (14), (15) and (16) it follows that $G_{12}, G_{13}, \ldots, G_{n-3\,n-2}, H_1, \ldots, H_{n-2} \in I(V)$.

Proposition 2. *The variety V is set-theoretically defined by*

$$F_1 = F_2 = F_3 = H_1 = 0 \quad \textit{if } n = 3,$$

and by

$$F_1 = \cdots = F_n = G_{12} = \cdots = G_{n-3\,n-2} = H_1 = \cdots = H_{n-2} = 0 \quad \textit{if } n \geqslant 4.$$

Proof. Suppose that $\mathbf{x} = (\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_n) \in K^{2n}$ fulfills the system of equations given in the claim. Set $u_i = \bar{x}_i$ for all $i = 1, \ldots, n-2$. As in the proof of Proposition 1, $F_{n-1}(\mathbf{x}) = 0$ implies that there is $u_n \in K$ such that $\bar{x}_n = u_n^a$ and $\bar{y}_{n-1} = u_n^d$. Let $v_{n-1} \in K$ be such that $\bar{x}_{n-1} = v_{n-1}^{p^\ell}$. We show that, up to replacing v_{n-1} with another p^ℓth root u_{n-1} of $\bar{x}_{n-1}$, $\mathbf{x}$ fulfills the parametrization of V. If $\bar{x}_{n-1} = 0$, then, for all $i = 1, \ldots, n-2$, and for $i = n$, $F_i(\mathbf{x}) = 0$ implies that $\bar{y}_i = 0$. Hence $u_{n-1} = 0$ yields the required representation of $\mathbf{x}$. So suppose that $\bar{x}_{n-1} \neq 0$. Let i be any index with $1 \leqslant i \leqslant n-2$. Condition $F_i(\mathbf{x}) = 0$ implies that $\bar{y}_i^{p^\ell} = u_i^{p^\ell b_i} v_{n-1}^{p^\ell c_i}$, i.e.,

$$\bar{y}_i = u_i^{b_i} v_{n-1}^{c_i} \omega_i \tag{17}$$

for some $\omega_i \in K$ such that

$$\omega_i^{p^\ell} = 1. \tag{18}$$

Now let $\bar{\eta}$ be a primitive p^ℓth root of 1. There is an integer z_i such that $\omega_i = \bar{\eta}^{z_i}$. On the other hand, by assumption (I) there is an integer w_i such that $c_i w_i \equiv z_i \pmod{p^\ell}$. Set $\eta = \bar{\eta}^{w_i}$. Then

$$\omega_i = \eta^{c_i}. \tag{19}$$

If $n=3$, then the only index i to be considered is $i=1$, and setting $u_{n-1}=v_{n-1}\eta$ will give

$$\bar{y}_1=u_1^{b_1}u_{n-1}^{c_1}.$$

Now let $n\geqslant 4$. We show that the same choice of parameter u_{n-1} yields the required representation for all $\bar{y}_1,\dots,\bar{y}_{n-2}$. This is trivially true if $u_i=0$ (i.e., $\bar{y}_i=0$) for all $i=1,\dots,n-2$, or $u_i=\bar{y}_i=0$ for all but one of these indices. So suppose that, for two indices i,j, with $1\leqslant i<j\leqslant n-2$, we have $u_i\neq 0$ and $u_j\neq 0$, which, under our present assumption that $u_{n-1}\neq 0$, is equivalent to $\bar{y}_i\neq 0$ and $\bar{y}_j\neq 0$. Then, by (17), $G_{ij}(\mathbf{x})=0$ implies that

$$u_i^{b_ig_{ij}}v_{n-1}^{c_ig_{ij}}\omega_i^{g_{ij}}u_j^{b_jg_{ji}}v_{n-1}^{c_jg_{ji}}\omega_j^{g_{ji}}=u_i^{b_ig_{ij}}u_j^{b_jg_{ji}}v_{n-1}^{r_{ij}p^{m_{ij}}}. \tag{20}$$

If we consider (14) and cancel equal terms on both sides of (20), we obtain

$$\omega_i^{g_{ij}}\omega_j^{g_{ji}}=1. \tag{21}$$

Now (19) and (21) imply that

$$\eta^{c_ig_{ij}}=\omega_j^{-g_{ji}}. \tag{22}$$

On the other hand, by (14) we have

$$c_ig_{ij}\equiv -c_jg_{ji}\pmod{p^{\ell}},$$

so that

$$\eta^{c_ig_{ij}}=\eta^{-c_jg_{ji}},$$

i.e., by (22),

$$\omega_j^{g_{ji}}=\eta^{c_jg_{ji}}.$$

Since p does not divide g_{ji}, there is an integer q such that $g_{ji}q\equiv 1 \pmod{p^{\ell}}$. Therefore, by (18) and (19), applied to the index j,

$$\omega_j=\omega_j^{g_{ji}q}=\eta^{c_jg_{ji}q}=\eta^{c_j}. \tag{23}$$

Note that (19), together with (23), implies that $\omega_i=\eta^{c_i}$ holds for all indices $i=1,\dots,n-2$ such that $\bar{y}_i\neq 0$. Set $u_n=v_n\eta$. Then, in view of (17), for all these indices we have

$$\bar{y}_i=u_i^{b_i}u_{n-1}^{c_i}. \tag{24}$$

This obviously also holds when $u_i=0$. It remains to show that $\bar{y}_n$ has the required form. This is certainly true if $\bar{x}_n=0$: then $u_n=0$, and $F_n(\mathbf{x})=0$ implies that $\bar{y}_n=0$.

So suppose that $\bar{x}_n\neq 0$. Condition $F_n(\mathbf{x})=0$ implies that $\bar{y}_n^{p^{\ell}}=u_{n-1}^{p^{\ell}}u_n^{p^{\ell}}$, i.e.,

$$\bar{y}_n=u_{n-1}u_n\omega, \tag{25}$$

for some $\omega \in K$ such that

$$\omega^{p^\ell} = 1. \tag{26}$$

If $\bar{x}_i = 0$ (i.e., $u_i = 0$) for all $i = 1, \dots, n-2$, replace u_{n-1} with $u_{n-1}\omega$. This will produce in (25) the required representation for $\bar{y}_n$, and will not affect the remaining entries of $\mathbf{x}$. So assume that $\bar{x}_j \neq 0$ for some index j, $1 \leqslant j \leqslant n-2$. We have that $H_j(\mathbf{x}) = 0$, together with (24) and (25), implies that

$$u_j^{b_j h_j} u_{n-1}^{c_j h_j} u_{n-1}^{k_j} u_n^{k_j} \omega^{k_j} = u_j^{b_j h_j} u_{n-1}^{r_j p^{m_j}} u_n^{a s_j} u_n^{d t_j}.$$

In view of (15) and (16), applied to the index j, if we cancel equal terms on both sides of the above equation, we obtain

$$\omega^{k_j} = 1. \tag{27}$$

Since p does not divide k_j, (26) and (27) imply that

$$\omega = 1.$$

Thus by (25),

$$\bar{y}_n = u_{n-1} u_n.$$

This completes the proof. □

3. The binomial arithmetical rank

We have just proven that V can always be set-theoretically defined by 4 binomial equations if $n = 3$ and by $2n - 2 + \binom{n-2}{2}$ binomial equations if $n \geqslant 4$. We now show that these numbers cannot be made smaller if $\operatorname{char} K \neq p$.

Proposition 3. *If* $\operatorname{char} K \neq p$, *then*

$$\operatorname{bar} V = \begin{cases} 4 & \text{if } n = 3, \\ 2n - 2 + \binom{n-2}{2} & \text{if } n \geqslant 4. \end{cases}$$

Proof. In view of the above remark, we only have to prove the inequality $\geqslant$. Let $\mathcal{B}$ be a set of binomials such that V is set-theoretically defined by the vanishing of all elements of $\mathcal{B}$. Of course we may assume that all elements of $\mathcal{B}$ are irreducible. Since, by Hilbert's Nullstellensatz, for all $i = 1, \dots, n-2$, F_i belongs to the radical of the ideal generated by $\mathcal{B}$ in R, one binomial of $\mathcal{B}$ is monic in y_i. By Lemma 2 it follows that the binomials $F_1, \dots, F_{n-1}$ defined in (2) and (3) belong to $\mathcal{B}$, together with some binomial F'_n monic in y_n. Let i be any index with $1 \leqslant i \leqslant n-2$, and let η be a primitive p^ℓth root of unity. Consider $\mathbf{x} = (\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_n) \in K^{2n}$, where $\bar{x}_i = \bar{x}_{n-1} = \bar{x}_n = 1$, $\bar{y}_i = \eta$ and $\bar{y}_{n-1} = \bar{y}_n = 1$, whereas the remaining entries are zero. Then $F_i(\mathbf{x}) = F'_n(\mathbf{x}) = 0$. Suppose for a contradiction that every $B \in \mathcal{B} \setminus \{F_i, F'_n\}$ has a monomial M such that

(a) $\mathrm{supp}(M) \subset \{x_{n-1}, x_n, y_{n-1}, y_n\}$, or
(b) $\mathrm{supp}(M) \not\subset \{x_i, x_{n-1}, x_n, y_i, y_{n-1}, y_n\}$.

Let M' be the other monomial of such a binomial B. In case (a), by Lemma 1(i) it follows that $\mathrm{supp}(M') \subset \{x_{n-1}, x_n, y_{n-1}, y_n\}$, so that $M(\mathbf{x}) = M'(\mathbf{x}) = 1$, and, consequently, $B(\mathbf{x}) = 0$. In case (b), we have that, for some index $k \neq i$ with $1 \leqslant k \leqslant n-2$, M is divisible by x_k or y_k. By Lemma 1(i) the same is true for M'. Since $\bar{x}_k = \bar{y}_k = 0$, we have that $M(\mathbf{x}) = M'(\mathbf{x}) = 0$, and thus $B(\mathbf{x}) = 0$. Therefore we have that, in any case, $B(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathcal{B}$. We show that, however, $\mathbf{x} \notin V$. If $\mathbf{x} \in V$, then $\mathbf{x}$ would fulfill the parametrization of V for a suitable choice of parameters $u_1, \ldots, u_n$. Then the arguments developed in the proof of Proposition 2 allow us to conclude that necessarily $u_i = \bar{x}_i$, $u_n = 1$, and $u_{n-1} = \eta^k$ for some integer k. Consequently, we would have

$$\eta = \bar{y}_i = u_i^{b_i} u_{n-1}^{c_i} = \eta^{kc_i},$$
$$1 = \bar{y}_n = u_n u_{n-1} = \eta^k,$$

which implies $\eta = 1$, against the definition of η. This shows that, for all $i = 1, \ldots, n-2$, there is a binomial $H_i' \in \mathcal{B} \setminus \{F_i, F_n'\}$ such that none of its monomials fulfills (a) or (b), i.e., both its monomials are of the form $x_i^{\alpha_i} x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} y_i^{\beta_i} y_{n-1}^{\beta_{n-1}} y_n^{\beta_n}$, with $\alpha_i > 0$ or $\beta_i > 0$. From Lemmas 1 and 2 it follows that the remaining exponents are not all zero. This suffices to prove the first case of the claim: since $\{F_1, F_2, F_3', H_1'\} \subset \mathcal{B}$, we have

$$\text{for } n = 3, \quad |\mathcal{B}| \geqslant 4.$$

Now suppose that $n \geqslant 4$. Note that, by construction, the H_i' are $n-2$ pairwise distinct binomials. Let i, j be indices such that $1 \leqslant i < j \leqslant n-2$. There are integers d_{ij}, d_{ji} such that

$$c_i d_{ij} - c_j d_{ji} = \gcd(c_i, c_j). \tag{28}$$

Consider $\mathbf{x} \in K^{2n}$ where $\bar{x}_i = \bar{x}_j = \bar{x}_{n-1} = 1$ and $\bar{y}_i = \eta^{d_{ij}}$, $\bar{y}_j = \eta^{d_{ji}}$, whereas the remaining entries are zero. Then $F_i(\mathbf{x}) = F_j(\mathbf{x}) = 0$. Suppose, for a contradiction, that for all $B \in \mathcal{B} \setminus \{F_i, F_j\}$, the support of neither monomial of binomial B is contained in $\{x_i, x_j, y_i, y_j\}$. Let B be any such monomial, and let M, M' be its monomials. Then, in view of Lemma 1, up to interchanging M and M', we have one of the following cases:

(a) for some index k with $1 \leqslant k \leqslant n-2$, $k \neq i, j$, M is divisible by x_k, and M' is divisible by y_k;
(b) M is divisible by one of the indeterminates x_n, y_{n-1}, y_n, and M' is divisible by one of the remaining two;
(c) M is divisible by x_{n-1}; in this case, by Lemma 1(iii), M' is divisible by one of the indeterminates $y_1, \ldots, y_{n-2}, y_n$; since $\mathrm{supp}(M') \not\subset \{y_i, y_j\}$, this takes us back to case (a) or (b).

In all the above cases $M(\mathbf{x}) = M'(\mathbf{x}) = 0$. We conclude that $B(\mathbf{x}) = 0$ for all $B \in \mathcal{B}$. Once again we show that assuming $\mathbf{x} \in V$ leads to a contradiction. In fact, under this assumption, $\mathbf{x}$ fulfills the parametrization of V with $u_i = u_j = 1$ and with $u_{n-1} = \eta^k$ for some integer k. Consequently,

$$u_i^{b_i} u_{n-1}^{c_i} = \eta^{kc_i}, \qquad u_j^{b_j} u_{n-1}^{c_j} = \eta^{kc_j}.$$

Hence $\bar{y}_i = u_i^{b_i} u_{n-1}^{c_i}$ and $\bar{y}_j = u_j^{b_j} u_{n-1}^{c_j}$ imply

$$\eta^{d_{ij}} = \eta^{kc_i}, \qquad \eta^{d_{ji}} = \eta^{kc_j},$$

which is equivalent to

$$d_{ij} \equiv kc_i \pmod{p^\ell}, \qquad d_{ji} \equiv kc_j \pmod{p^\ell}.$$

Therefore

$$c_j d_{ij} \equiv c_i d_{ji} \pmod{p^\ell},$$

which is incompatible with (28), since, in view of assumption (I), p does not divide $\gcd(c_i, c_j)$. This shows that $\mathbf{x} \notin V$ and provides the required contradiction. We conclude that $\mathcal{B}$ must contain, for all indices i, j such that $1 \leqslant i < j \leqslant n-2$, a binomial G'_{ij}, other than F_i, F_j, such that one of its monomials is of the form $x_i^{\gamma_i} x_j^{\gamma_j} y_i^{\delta_i} y_j^{\delta_j}$. By Lemma 1(i), Lemma 2 and irreducibility, it follows that one of γ_i, δ_i and one of γ_j, δ_j are positive. It follows that the G'_{ij} are $\binom{n-2}{2}$ pairwise distinct binomials. It is also evident that the sets $\{F_1, \ldots, F_{n-1}, F'_n\}$, $\{G_{12}, \ldots, G_{n-2\,n-3}\}$ and $\{H'_1, \ldots, H'_{n-2}\}$ are pairwise disjoint. Therefore

$$\text{for } n \geqslant 4, \quad |\mathcal{B}| \geqslant n + \binom{n-2}{2} + n - 2 = 2n - 2 + \binom{n-2}{2}.$$

This completes the proof. □

4. A lower bound for the arithmetical rank

In this section we give a lower bound for $\operatorname{ara} V$ when $\operatorname{char} K \neq p$. We will use the following result, which is due to Newstead and is quoted from [6]. It is based on étale cohomology (H_{et}). We refer to [10] or to [11] for the basic notions.

Lemma 3. *Let $W \subset \tilde{W} \subset K^N$ be affine varieties. Let $d = \dim \tilde{W} \setminus W$. If there are s polynomials $F_1, \ldots, F_s$ such that $W = \tilde{W} \cap V(F_1, \ldots, F_s)$, then*

$$H_{\text{et}}^{d+i}(\tilde{W} \setminus W, \mathbb{Z}/r\mathbb{Z}) = 0 \quad \textit{for all } i \geqslant s$$

and for all $r \in \mathbb{Z}$ which are prime to $\operatorname{char} K$.

We prove the following result.

Proposition 4. *If* $\operatorname{char} K \neq p$, *then* $\operatorname{ara} V \geqslant 2n - 2$.

Proof. We have to show that V cannot be defined set-theoretically by $2n-3$ equations. According to Lemma 3 this is true if

$$H_{\text{et}}^{4n-3}\big(K^{2n} \setminus V, \mathbb{Z}/p\mathbb{Z}\big) \neq 0.$$

Since $K^{2n} \setminus V$ is a Zariski open subset of an affine space, and is therefore nonsingular by Poincaré Duality (see [10, Corollary 11.2, p. 276]), this is equivalent to

$$H_c^3\big(K^{2n} \setminus V, \mathbb{Z}/p\mathbb{Z}\big) \neq 0, \tag{29}$$

where H_c denotes cohomology with compact support. For the sake of simplicity, we shall henceforth omit the coefficient group $\mathbb{Z}/p\mathbb{Z}$. According to [10, Remark 1.30, p. 94], there is an exact sequence:

$$H_c^2\big(K^{2n}\big) \to H_c^2(V) \to H_c^3\big(K^{2n} \setminus V\big) \to H_c^3\big(K^{2n}\big). \tag{30}$$

Recall that

$$H_c^i\big(K^m\big) \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{for } i = 2m, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

See [11, Example 16.3, pp. 98–99], together with Poincaré Duality, for a proof. Hence the left and the right group in (30) are zero, so that the middle map is a group isomorphism. Thus our claim (29) is equivalent to

$$H_c^2(V) \neq 0. \tag{32}$$

The proof of (32) needs some preparation. Consider the following morphism of schemes:

$$\begin{aligned} &\varphi : K^n \to V \\ &(u_1, \ldots, u_n) \mapsto \big(u_1, \ldots, u_{n-2}, u_{n-1}^{p^\ell}, u_n^a, u_1^{b_1} u_{n-1}^{c_1}, \ldots, u_{n-2}^{b_{n-2}} u_{n-1}^{c_{n-2}}, u_n^d, u_{n-1}u_n\big). \end{aligned}$$

Let W be the subvariety of K^n defined by $u_1u_{n-1} = \cdots = u_{n-2}u_{n-1} = u_n = 0$. We show that φ induces by restriction an isomorphism of schemes:

$$\tilde{\varphi} : K^n \setminus W \to V \setminus \varphi(W).$$

Note that $V \setminus \varphi(W)$ is the union of the open subsets

$$V_i = \{\mathbf{x} \in V \mid y_i \neq 0\} \quad (i = 1, \ldots, n-1).$$

Moreover, for all $\mathbf{x} \in V$ and all $i = 1, \ldots, n-2$,

$$y_i \neq 0 \quad \text{is equivalent to} \quad x_i \neq 0 \quad \text{and} \quad x_{n-1} \neq 0,$$

and

$$y_{n-1} \neq 0 \quad \text{is equivalent to} \quad x_n \neq 0.$$

Thus we have

$$U_i = \varphi^{-1}(V_i) = \big\{\mathbf{u} \in K^n \mid u_i u_{n-1} \neq 0\big\} \quad (i = 1, \ldots, n-2),$$

and

$$U_{n-1}=\varphi^{-1}(V_{n-1})=\{\mathbf{u}\in K^n \mid u_n\neq 0\}.$$

By assumption (II) of Section 1 there are integers r, s such that $ar+ds=1$; by assumption (I) there are, for all $i=1,\dots,n-2$, integers v_i, w_i such that $c_iv_i+p^\ell w_i=1$. The following morphisms of schemes are inverse to the restrictions of φ to U_i:

$$\begin{aligned}
&V_i\to U_i \quad (i=1,\dots,n-2),\\
&(x_1,\dots,x_n,y_1,\dots,y_n)\mapsto\left(x_1,\dots,x_{n-2},\frac{y_i^{v_i}x_{n-1}^{w_i}}{x_i^{b_iv_i}},\frac{y_nx_i^{b_iv_i}}{y_i^{v_i}x_{n-1}^{w_i}}\right),\\
&V_{n-1}\to U_{n-1},\\
&(x_1,\dots,x_n,y_1,\dots,y_n)\mapsto\left(x_1,\dots,x_{n-2},\frac{y_n}{x_n^ry_{n-1}^s},x_n^ry_{n-1}^s\right).
\end{aligned}$$

We have just proven that φ is an isomorphism. Hence, for all indices i, it induces an isomorphism in cohomology with compact support:

$$\varphi_i: H_c^i(V\setminus\varphi(W))\xrightarrow{\simeq}H_c^i(K^n\setminus W). \tag{33}$$

Now consider the subvariety Y of W defined by $u_1=\dots=u_{n-2}=u_n=0$. Then Y can be identified with K, and $W\setminus Y$ with the set of all points in K^{n-1} such that $u_{n-1}=0$, whereas not all u_i with $1\leqslant i\leqslant n-2$ are zero; in other words, $W\setminus Y$ can be identified with $K^{n-2}\setminus\{0\}$. It can be easily seen that also $\varphi(Y)$ (which is a closed subset of $\varphi(W)$) and $\varphi(W)\setminus\varphi(Y)$ can be identified with K and $K^{n-2}\setminus\{0\}$, respectively. From (31) and the long exact sequence in [10, Remark 1.30, p. 94], it easily follows that

$$H_c^i(K^m\setminus\{0\})\simeq\begin{cases}\mathbb{Z}/p\mathbb{Z} & \text{for } i=1,2m,\\ 0 & \text{otherwise.}\end{cases} \tag{34}$$

Now, in view of the above identifications, (31) and (34) imply

$$H_c^i(\varphi(Y))\simeq H_c^i(Y)\simeq\begin{cases}\mathbb{Z}/p\mathbb{Z} & \text{for } i=2,\\ 0 & \text{otherwise,}\end{cases} \tag{35}$$

and

$$H_c^i(\varphi(W)\setminus\varphi(Y))\simeq H_c^i(W\setminus Y)\simeq\begin{cases}\mathbb{Z}/p\mathbb{Z} & \text{for } i=1,2n-4,\\ 0 & \text{otherwise.}\end{cases} \tag{36}$$

In the sequel we shall use the following exact sequences:

$$H_c^i(K^n)\to H_c^i(W)\to H_c^{i+1}(K^n\setminus W)\to H_c^{i+1}(K^n), \tag{37}$$

$$H_c^{i-1}(Y)\to H_c^i(W\setminus Y)\to H_c^i(W)\to H_c^i(Y)\to H_c^{i+1}(W\setminus Y). \tag{38}$$

There is a similar sequence obtained from (38) by replacing W and Y with $\varphi(W)$ and $\varphi(Y)$, respectively. We are now ready to prove claim (32). Consider sequence (38) for $i=2$. We have

$$\begin{tikzcd} H_c^1(Y) \arrow[r] \arrow[d, equal] & H_c^2(W\setminus Y) \arrow[r] & H_c^2(W) \arrow[r] & H_c^2(Y) \arrow[r] \arrow[d, "\wr"] & H_c^3(W\setminus Y) \arrow[d, equal] \\ 0 & & & \mathbb{Z}/p\mathbb{Z} & 0, \end{tikzcd} \tag{39}$$

where the equalities and the isomorphisms follow from (35) and (36). From (36) we also have that

$$H_c^2(W\setminus Y)\simeq\begin{cases}\mathbb{Z}/p\mathbb{Z} & \text{if } n=3,\\ 0 & \text{if } n\geqslant 4.\end{cases}$$

In view of (39) we deduce that

$$\left|H_c^2(W)\right|=\begin{cases}p^2 & \text{if } n=3,\\ p & \text{if } n\geqslant 4.\end{cases}$$

A similar result holds for $H_c^2(\varphi(W))$. Consequently,

$$\left|H_c^2\big(\varphi(W)\big)\right|=\left|H_c^2(W)\right|\neq 0. \tag{40}$$

On the other hand, from the exact sequence (37), for $i=2$, we have

$$\begin{tikzcd} H_c^2\big(K^n\big) \arrow[r] \arrow[d, equal] & H_c^2(W) \arrow[r] & H_c^3\big(K^n\setminus W\big) \arrow[r] & H_c^3\big(K^n\big) \arrow[d, equal] \\ 0 & & & 0, \end{tikzcd}$$

where the equalities are a consequence of (31). We thus have an isomorphism:

$$H_c^2(W)\xrightarrow{\simeq}H_c^3\big(K^n\setminus W\big). \tag{41}$$

Next we show that the following map, induced in cohomology by the restriction of φ to W,

$$\varphi_2'\colon H_c^2\big(\varphi(W)\big)\to H_c^2(W)$$

is not injective. It is well known that the restriction of φ to Y

$$\begin{aligned}\varphi_|\colon Y&\to\varphi(Y),\\ u_{n-1}&\mapsto u_{n-1}^{p^\ell},\end{aligned}$$

induces multiplication by p^ℓ in cohomology (see [11, Remark 24.2(f), p. 135]). Thus φ gives rise to the following commutative diagram:

$$\begin{array}{ccccc}
 & & \mathbb{Z}/p\mathbb{Z} & & \\
 & & \Big| \wr & & \\
H_c^2(W) & \xrightarrow{\alpha} & H_c^2(Y) & \longrightarrow & 0 \\
\Big\uparrow \varphi_2' & & \Big\uparrow \cdot p^\ell & & \\
H_c^2(\varphi(W)) & \longrightarrow & H_c^2(\varphi(Y)) & \longrightarrow & 0 \\
 & & \Big| \wr & & \\
 & & \mathbb{Z}/p\mathbb{Z}, & &
\end{array}$$

where the first row is part of the exact sequence (39) and the second row is similarly derived from (38). Note that multiplication by p^ℓ is the zero map. If φ_2' were injective, in view of (40) it would also be surjective. But then so would be $\alpha\varphi_2'$; this map, however, because of commutativity, is the zero map, which is a contradiction.

Finally consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & H_c^2(W) & \xrightarrow{\simeq} & H_c^3(K^n \setminus W) \\
 & & \Big\uparrow \varphi_2' & & \Big\uparrow \varphi_3 \ \wr| \\
H_c^2(V) & \longrightarrow & H_c^2(\varphi(W)) & \xrightarrow{\beta} & H_c^3(V \setminus \varphi(W)),
\end{array}$$

where the isomorphisms are those given in (33) and (41). Since φ_2' is not injective, nor is β. It follows that $H_c^2(V) \neq 0$, i.e., (32) is true. This completes the proof. □

In the special case where $n = 3$ Proposition 4 yields ara $V \geqslant 4$; on the other hand, by Proposition 3, we have bar $V = 4$. Thus ara $V =$ bar $V = 4$. Moreover, according to a classical theorem, proven by Eisenbud and Evans [7], and, independently, by Storch [12], every variety in the N-dimensional affine space can be defined by a system of N equations. Thus the above results can be summarized as follows:

Corollary 2. *It holds*:

(i) ara $V = n$, *if* char $K = p$;
(ii) $2n - 2 \leqslant$ ara $V \leqslant 2n$, *if* char $K \neq p$.

In (i), ara V *defining equations can be chosen to be binomial, whereas this is possible in* (ii) *if and only if* $n = 3$, *and in this case* ara $V = 4$.

The problem of determining ara V when char $K \neq p$ and $n \geqslant 4$ remains open.

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